

## LETTER TO THE EDITOR

**A cellular automaton with two phase transitions****György Szabó and István Borsos**

Research Institute for Technical Physics and Materials Science, PO Box 49, H-1525 Budapest, Hungary

Received 21 January 2002

Published 22 March 2002

Online at [stacks.iop.org/JPhysA/35/L189](http://stacks.iop.org/JPhysA/35/L189)**Abstract**

We have reinvestigated a one-dimensional stochastic cellular automaton introduced previously to study a kinetic critical phenomenon belonging to the parity-conserving directed percolation universality class. It is found that this model exhibits a second phase transition which is related to an ordering process resulting in the extinction of domains of odd numbers of 1s or 0s.

PACS numbers: 05.65.+b, 05.10.Gg, 05.40.–a, 64.60.Cn, 64.60.Ht

Several years ago Grassberger *et al* [1] introduced two one-dimensional stochastic cellular automata called models (A) and (B), which undergo the same kinetic phase transition when varying the stochastic parameter  $P$ . This critical transition belongs to the parity-conserving directed percolation universality class [2]. We have tried to study this critical behaviour by fitting Padé approximants to the results obtained for a dynamical cluster method for different cluster sizes. Our attempts failed in both cases although this method was very efficient for other models [3,4]. When analysing the failure we observed a weak discrepancy in the deterministic limit ( $P \rightarrow 1$ ) for both models. Henceforth we concentrate on model (B) exhibiting an additional ordering process.

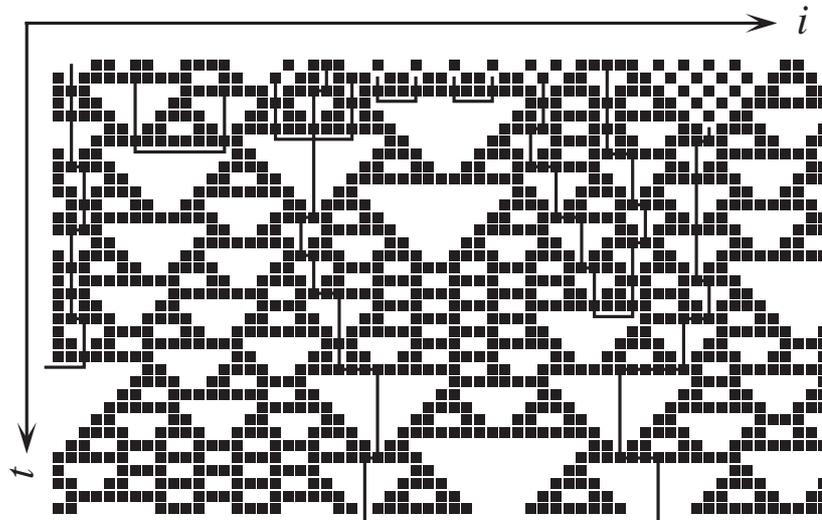
In the present one-dimensional stochastic cellular automaton [1] the state is described by a set of variables  $s_i(t) = 0$  or  $1$  ( $i \in \mathbb{Z}$ ) at a given discrete time  $t$ . In the subsequent time steps the new state  $s_i(t+1)$  is determined by the states  $s_{i-1}(t)$ ,  $s_i(t)$  and  $s_{i+1}(t)$ . The translation-invariant rules are characterized by the following transitions:

$$\begin{array}{rcccccccc} t: & 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ t+1: & 0 & s & 1 & 1 & s & 0 & 1 & 0 \end{array}$$

where

$$s = \begin{cases} 1 & \text{with probability } P, \\ 0 & \text{with probability } (1 - P). \end{cases} \quad (1)$$

Starting this system from a random initial configuration one can observe different pattern evolutions depending on the value of  $P$ . For  $P = 0$  the system tends toward a stationary



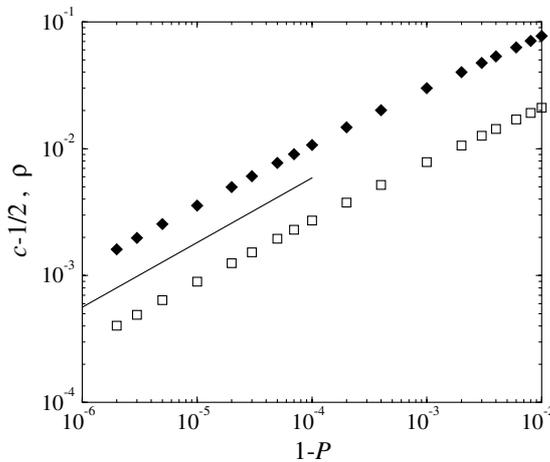
**Figure 1.** Pattern evolution created from a random initial configuration (top row) for  $P = 1$ . Each closed square represents the state 1. The motion of the centre of domains of odd length is indicated by solid lines.

state where the ‘chessboard’ and ‘anti-chessboard’ phases are separated by standing kinks. For small  $P$ -values the kinks walk randomly and two kinks annihilate each other if they meet. Due to this annihilating random walk of kinks, the system tends toward an absorbing (ordered) state without kinks if  $P < P_{cr} = 0.555(1)$ . Above this threshold value ( $P > P_{cr}$ ) the kink density remains finite in the stationary state because some particular random events produce three kinks from one. The kink density characterized by the probability of finding  $(1, 1)$  pairs in two neighbouring sites shows a power-law behaviour (i.e.  $p_2(1, 1) \simeq (P - P_{cr})^\beta$ ) in the vicinity of  $P_{cr}$  if the system size goes to infinity. In this case the parity of the number of kinks is conserved for both the annihilation and kink production. Unlike in the traditional branching annihilating random walks, here the very likely mutual annihilation of the parent and one of its offspring kinks does not leave an empty site behind; therefore the observed critical transition differs from the robust directed percolation universality class [1, 2, 5].

The kink density increases with  $P$  until it reaches its maximum at  $P \simeq 0.99[\max p_2(1, 1) = 0.3814]$ . Surprisingly, after this maximum,  $p_2(1, 1)$  shows an anomalous decrease toward  $p_2(1, 1) = 0.375$  with a slope diverging when  $P \rightarrow 1$ . Notice that the magnitude of this decrease is less than 2% of the maximal value. Our analysis is focused on this weak discrepancy.

For  $P = 1$ , the present cellular automaton is equal to the rule number 122 in the notation of Wolfram [6]. In this limit case the chessboard structures are eliminated by the  $(1, 1)$  pairs on their boundaries appearing sooner or later during the evolution as shown in figure 1. At the same time this spatio-temporal pattern indicates a particular ordering process yielding the extinction of domains with odd number of 1s (or 0s). Henceforth the centre of an odd-length domain is considered as a particle whose density is characterized by  $\rho$ . Figure 1 illustrates two such particles annihilating each other when they meet on the upper side of an empty triangle.

Finally all the particles die out and the stationary state can be built up from  $(1, 1)$  and  $(0, 0)$  pairs. In this subset of phase space the dynamical rule becomes equivalent to the 6/16 rule, as happens for some other cellular automata [4, 7, 8]. Straightforward analytical calculation yields that the stationary state is equivalent to a random series of  $(1, 1)$  and  $(0, 0)$  pairs. Consequently,



**Figure 2.** A log-log plot of  $c - 1/2$  (open squares) and  $\rho$ , the density of domains with odd length (diamonds), versus  $1 - P$ . The slope of the theoretical prediction is plotted as a solid line.

the average densities of 1s and 0s are equal ( $p_1(1) = p_1(0) = 1/2$ ), while  $p_2(1, 1) = 3/8$ . This uncorrelated distribution yields that the above-mentioned particles perform random walks if their density is low.

In the vicinity of this deterministic limit, particle pairs are created with a probability  $1 - P$ . Consequently, the present system can be considered as a set of annihilating particles walking randomly with an external source creating particle pairs. Rácz [9] has recognized that this system can be mapped to the one-dimensional Ising model for zero magnetic field. In the stationary state the exact solution gives an average density of particles  $\rho \simeq \sqrt{1 - P}$  in the limit  $P \rightarrow 0$ .

The consequences of the above prediction were checked by Monte Carlo simulations. The simulations were performed on a lattice for different  $P$ -values and sizes (up to  $L = 2 \times 10^6$ ) under periodic boundary conditions. During the simulations we monitored the number of 1s and odd-length domains. After some suitable transient the stationary values were determined by averaging over a sampling time that exceeded  $5 \times 10^5$  Monte Carlo steps per site in the vicinity of  $P = 1$ .

Figure 2 compares the Monte Carlo data with the above theoretical prediction for  $\rho$ . The fitted slope (0.49(2)) agrees very well with the theoretical value (1/2). Furthermore, the concentration of extra 1s ( $c - 1/2$ ) is proportional to the density of odd-length domains.

It is also found that the variation of the probability of extra (1, 1) pairs ( $p_2(1, 1) = 3/8$ ) exhibits similar  $P$ -dependence in this limit. This nonanalytical behaviour at  $P = 1$  has caused the failure of above-mentioned Padé approximation based on the series expansion with respect to  $1 - P$ .

Here it is worth mentioning that for the other model (A) introduced by Grassberger *et al* [1] we have also observed some discrepancy in the limit  $P \rightarrow 1$ . The detailed numerical analysis, however, has confirmed that in this case the anomalous decrease of kink density becomes linear if  $P \rightarrow 1$ . We suspect that this behaviour is related to the appearance of those complex configurations which are prohibited in the deterministic limit [10].

In summary, simple stochastic cellular automata can exhibit two different ordering processes. In this case, difficulties can emerge in investigating the dominant one by sophisticated techniques although the weaker process yields only a very small effect.

**References**

- [1] Grassberger P, Krause F and von der Twer T 1984 *J. Phys. A: Math. Gen.* **17** L105
- [2] Grassberger P 1989 *J. Phys. A: Math. Gen.* **22** L1107
- [3] Dickman R, Wang J-S and Jensen I 1991 *J. Chem. Phys.* **94** 8252
- [4] Szabó G and Ódor G 1994 *Phys. Rev. E* **49** 2764
- [5] Cardy J L and Täuber U C 1998 *J. Stat. Phys.* **90** 1
- [6] Wolfram S 1983 *Rev. Mod. Phys.* **55** 601
- [7] Grassberger P 1983 *Phys. Rev. A* **28** 3666
- [8] Eloranta K and Nummelin E 1992 *J. Stat. Phys.* **69** 1131
- [9] Rácz Z 1985 *Phys. Rev. Lett.* **55** 1707
- [10] Badii R and Politi A 1997 *Phys. Rev. Lett.* **78** 444