Vortex dynamics in a three-state model under cyclic dominance

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The evolution of domain structure is investigated in a two-dimensional voter model with three states under cyclic dominance. The study focuses on the dynamics of vortices, defined by the points where the three states (domains) meet. We can distinguish vortices and antivortices which walk randomly and annihilate each other. The domain wall motion can create vortex-antivortex pairs at a rate that increases due to cyclic dominance. This mechanism is contrasted with a branching annihilating random walk (BARW) in a particle-antiparticle system with density-dependent pair creation rate. Numerical estimates for the critical indices of the vortex density [β = 0.29(4)] and of its fluctuation [γ = 0.34(6)] improve an earlier Monte Carlo study [K. Tainaka and Y. Itoh, Europhys. Lett. 15, 399 (1991)] of the three-state cyclic model in two dimensions.

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The self-organizing domain structures in the cyclic variants of the Lotka-Volterra model [1,2] have been extensively investigated because similar spatiotemporal oscillations can appear in chemical reactions as well as in more complex ecological processes. These phenomena can be well studied within the formalism of voter models [3] we will follow henceforth. Different versions of the three-state voter models under cyclic dominance on a square lattice were introduced by Itoh and Tainaka [4–6]. Bramson and Griffeath [7] studied the flux and fixation in cyclic particle systems. The pattern formation in cyclic cellular automata was investigated by Fisch [8]. Using a pair approximation Frachebourg and Krapivsky [9] have shown that fixation occurs in a cyclic Lotka-Volterra model if the system is started from a random initial state and the number of states exceeds a critical value dependent on the dimension. According to this result, the system tends toward a self-organizing, inhomogeneous state if the number of states is less than 14 on a square lattice. Unfortunately, the theoretical understanding of this mechanism maintaining the inhomogeneous state is still incomplete.

In this work we consider the features of a self-organizing domain structure in a two-dimensional system using the concept of vortices defined for three-state models. Instead of studying the average sizes of the domains, Tainaka and Itoh [5] have determined the average density of vortices (c = ⟨Nv⟩/L2, where Nv denotes the number of vortices in a system with L×L lattice points). The vortices are points in these domain structures where the three different states (A, B, and C) and the three types of domain walls meet. Evidently, the value of c−1 represents roughly the average domain area (size). The investigation of vortex density was strongly motivated by the fact that its determination is much easier than the evaluation of the average domain size.

Figure 1 shows a (three-color) domain structure on the macroscopic scale. It is easy to recognize that two types of vortices may be distinguished as indicated by black and white bullets in the figure. We will call them vortex and antivortex depending on whether we find an ABC or ACB order when going clockwise around the center. A simple rule may be deduced at first glance, namely, the number of vortices is even around a closed domain. More precisely, the vortices and antivortices are alternately located along the closed boundary of a domain. This feature has serious consequences when the motion and collision of vortices is considered.

During the time evolution of a three-color domain structure the vortices move together with the boundaries. In these processes the vortices can collide and annihilate each other. Figure 2 illustrates the typical elementary events (after the transient processes) whose combinations describe all the possible phenomena related to the creation, annihilation, and collisions of the vortices. In the present approach the topological situations where four or more domain boundaries meet at a given point are considered as instantaneous events of collisions, pair annihilations, or creations. Evidently, the illustrated elementary events modify the connectivity among the vortices. This connectivity, however, can be modified by either fusion or fission of domains without any change in their constellation.

A numeric analysis of the vortex density was performed

FIG. 1. A typical domain structure for a three-state (color) model. The black bullets with white borders and the white bullets with black borders represent the vortices and antivortices.
by Tainaka and Itoh [5] in a two-dimensional voter model where the voters, located on a square lattice, should choose among three states (opinions): A, B, and C. The system evolution is governed by a simple algorithm: a randomly chosen voter or one of its nearest neighbors can modify their opinion if those are different. If the chosen voter and its neighbor are in the states A and B, then the first voter adopts opinion B with probability \( P \), otherwise its neighbor changes its state to A. This adoption rule is repeated cyclically for the B→C and C→A cases. Similar cyclic dominance characterizes the “paper, scissors, stone” games [10].

The direction of dominance can be reversed by replacing 1-\( P \) for \( P \), therefore the analyses are restricted to \( P > 1/2 \). For \( P = 1/2 \) the present model is equivalent to a traditional voter model [3], which exhibits a (three-color) domain coarsening phenomenon if initially the voter states are random. In other words, the finite system evolves into one of the three homogeneous states while the vortex density goes to zero.

For \( P > 1/2 \), however, a self-organizing domain structure is maintained, and the vortex density tends to a stationary value, \( N_v \), dependent on \( P \). More precisely, Tainaka and Itoh have found a power law behavior, namely,

\[
c \sim (P - P_c)^\beta,
\]

where \( P_c = 1/2 \) and \( \beta \sim 0.40 \) [5].

We have repeated these simulations using larger system size (\( 400 \times 400 \)) and longer sampling times. In the vicinity of \( P_c (P - P_c < 0.01) \) the thermalization is chosen to be longer than 250,000 MCS (Monte Carlo steps per particle). In the stationary state we have determined all the four-point configuration probabilities \( Q(n_1,n_2,n_3,n_4) \mid n_i = A,B, \text{ or } C; \ (i = 1,2,3,4) \) of a \( 2 \times 2 \) cluster. In our notation the \( (A,A,B,C) \) configuration refers to a vortex, the \( (A,A,C,B) \) to an antivortex, and the \( (A,B,C,A) \) configuration can be interpreted as a vortex-antivortex pair (before their annihilation or after their birth). Monitoring the vortex density we were able to determine its average value and its fluctuation \( \langle \chi = L^2 \langle (N_v/L^2 - c)^2 \rangle \rangle \) in the stationary states as a function of \( p = P - P_c \).

The results of our simulations are summarized in two log-log plots (see Figs. 3 and 4). Figure 3 demonstrates that we have reproduced Tainaka and Itoh’s data [5] for \( p > 0.1 \). In this region the fitted power law (dashed line) is characterized by an exponent \( \beta = 0.41(3) \) in agreement with Tainaka and Itoh. For smaller \( p \) values, however, we have found a different exponent \( \beta = 0.29(4) \) (solid line). Similar crossover behavior can be observed for the fluctuation as shown in Fig. 4. The fluctuation remains approximately constant for \( p > 0.1 \), whereas it follows a power law \( \chi \sim (P - P_c)^\gamma \) with \( \gamma = 0.34(6) \) for smaller \( p \) values.

The above-mentioned configuration probabilities satisfy the reflection, rotation of 90° and cyclic symmetries. Consequently, all these quantities are describable, within a cluster approximation, by introducing six independent parameters that can be evaluated by solving a set of equations of motion for the four-point configuration probabilities. Previously, this method has been proved to be a very efficient tool.
Vortices will then expand and become capable to create fur-
short time but, sooner or later, a vortex-antivortex pair will
the original vortex. Most of the pairs are annihilated within a
row ‘‘arms,’’ which is a favored situation for the pair cre-
antivortex pairs inside the spiral because it consists of nar-
series of snapshots in Fig. 5. Due to the randomness the
average time evolution is decorated by noise as shown in a
boundaries yields a spiral formation around the center. The
investigations of stochastic cellular automata [11],
evolutionary games [12], different lattice versions of the
Lotka-Volterra models (at the level of pair approximation)
[6,9], and for a two-dimensional driven lattice gas main-
taining a self-organizing domain structure, too [13]. In the
present case, the calculations show a very weak \( P \) de-
pendence of the configuration probabilities, including the vortex
constellations mentioned above, in contrast with the MC
simulations. At the level of the pair approximation the con-iguration probabilities are independent of \( P \) [6], while the
mean-field (one-point) approximation predicts homogeneous
oscillatory behavior. At the five-point level, on the other
hand, the preliminary calculations (integrating numerically
the equations of motion) also indicate a very weak \( P \)
dependence. This puzzling failure of the dynamical cluster tech-
nique inspired us to search for a mechanism observable at
macroscopic (or mesoscopic) scale.

For \( P = 1/2 \), the domain coarsening is accompanied by the
annihilation of pairs, and this process is not prevented by the
weak spontaneous pair creation. Under cyclic dominance
\( (P > 1/2) \), however, we have detected the appearance of a
pair creation mechanism which is able to compensate for the
previous annihilation process, yielding a finite vortex den-
sity. We have displayed in Fig. 5 the evolution of a single
vortex whose geometry allows us to recognize the essential
processes. Here, instead of the usual periodic boundary con-
ditions, we have assumed that a voter residing on the periphery
and its ‘‘hypothetic’’ outer neighbor have always the same
opinion. Along the boundaries one can observe a cyclic in-
vasion whose average velocity component perpendicular to
the border line is proportional to \( P - P_c \). This motion of the
boundaries yields a spiral formation around the center. The
average time evolution is decorated by noise as shown in a
series of snapshots in Fig. 5. Due to the randomness the
neighboring boundaries can contact and create vortex-
antivortex pairs inside the spiral because it consists of nar-ow ‘‘arms,’’ which is a favored situation for the pair cre-
ation. The created pairs can be considered as the offsprings
of the original vortex. Most of the pairs are annihilated within a
short time but, sooner or later, a vortex-antivortex pair will
eventually drift apart from each other. The corresponding
vortices will then expand and become capable to create fur-
ther offsprings via the same spiral formation process. The
self-organizing domain structure can thus be maintained by
this mechanism.

Within the framework of the vortex language, the evolu-
tion of domain structure can be described by the two-
dimensional motion of vortices, allowing the annihilation
and creation of pairs. Assuming that the motion of vortices is
dominantly controlled by noise, the present problem can be
considered as a so-called branching annihilating random
walk (BARW) with two types of particles created and anni-
hilated in pairs. The branching process is evidently con-
trolled by the value of \( P - P_c \), though the mathematical re-
lation between the branching rate and \( P - P_c \) is not yet
clarified. Furthermore, the branching rate is affected by the
nearest neighbor distances because it limits the spiral forma-
tion.

The traditional BARWs have been intensively studied in
the last years (for recent reviews, see the work by Cardy and
Täuber [14] and Marro and Dickman [15]) because they un-
dergo a critical transition when varying the rate of branching.
The corresponding critical behavior belongs to the directed
percolation (DP) universality class [16] involving the
Reggeon field theory [17], the surface reaction [18] and
Schlögl models [19], and the extinction phenomena observed in
spatial evolutionary games [12]. According to the ‘‘DP
conjecture’’ [20] most of the one-component model with a
single absorbing state belong to the DP universality class.
Exceptions can appear when additional symmetries [21] or
conservation laws are introduced. Well known examples are
those models in which the parity of particles is conserved
during the elementary processes [22,23]. The introduction of
two (or more) types of particles has enlarged the number of
possible universality classes [14,24].

The field theoretical results [14,24,25] indicate that mean-
field approaches can give satisfactory description of the two-
dimensional BARW models with two types of particles. This
observation motivated us to try to describe the branching
annihilating random walks of vortices and antivortices through
a mean-field equation with a density-dependent branching rate, namely,

\[
\frac{\partial}{\partial t} c = -c^2 + \lambda(c)c,
\]

where \( c \) denotes the concentration of vortices and antivorti-
ces. The first term describes the annihilation process whose
prefactor is eliminated by choosing a suitable time scale. For
simplicity we suppose that the density dependence of the
branching rate follows a power law with exponent \( \nu \),

\[
\lambda(c) = pc^{-\nu},
\]

where \( p = A(P - P_c) \). Notice that this branching rate
diverges in the limit \( c \rightarrow 0 \) if \( \nu > 0 \). In the stationary state one
can easily determine the concentration as a function of \( p \):

\[
c \sim p^{1/(1+\nu)}.
\]

Comparing this formula with the above MC results one can
conclude that the present mean-field description predicts \( \nu
= 1.5 \) if \( P - P_c > 0.1 \) and \( \nu = 2.5 \) for the smaller value of \( P
- P_c \).
In order to check the role of fluctuations we have performed MC simulations on a particle-antiparticle BARW model. The system evolution is governed by nearest neighbor jumps, particle-antiparticle pair annihilations, and creations as follows. A randomly chosen particle (or antiparticle) can create a particle-antiparticle pair located on two randomly chosen nearest neighbor sites with a probability $P_{br}$, otherwise this particle jumps to one of the nearest neighbor sites. The processes which would result in two particles (or antiparticles) residing on the same point are blocked. A particle-antiparticle pair is annihilated if they would stay on the same point as a result of the mentioned elementary processes. The branching rate $P_{br}$ is determined for a given particle as a product $P_{br} = pR_1R_2R_3$, where $R_1$ ($R_2$, $R_3$) denotes the distance between the chosen particles and its first (second, third) neighbor antiparticles. The choice of this branching rate is motivated by the topological fact that, in the original voter model, a vortex is connected directly to three antivortices by boundary lines (see Fig. 1). During the simulations the value of $P_{br}$ can become larger than 1 very rarely, thus, we did not need to reduce the time unit in which each particle has a chance to create offsprings or to jump. Initially particles and antiparticles with equal numbers are distributed randomly on a square lattice. The system size is varied from $L=100$ to 400 when decreasing $p$. During the simulations we have recorded the number of particles and determined the average value of the concentration $c$ and its fluctuation $\chi$ in the stationary state as defined above. From these MC results (see Fig. 6) we could confirm that the concentration follows a power law (solid line) with an exponent $\beta = 0.42(3)$. It is emphasized that this value of $\beta$ agrees very well with the prediction of the above mean-field formalism ($\beta_{MF} = 0.4$ for $\nu = 1.5$). This result implies that any value of the exponent $\beta$ is reproducible with the parameter adjustment of a more sophisticated BARW model.

Considering the fluctuations obtained by simulations, a striking difference is found between the present BARW model and the three-candidate voter model. The BARW simulations (squares in Fig. 6) indicate that $\chi(p) \approx c(p)$, which seems to be a typical behavior for the BARW models [25,26]. For the voter model, on the contrary, $\chi$ diverges (see Fig. 4) for small values of the control parameter. We can identify two possible sources for this discrepancy. First, during the diffusive motion the particle-antiparticle annihilation results in an aggregation of the two species [27], whereas in the vortex-antivortex system such a process is strongly limited by the mentioned topological features. Second, the threefold degeneracy of the absorbing states of the three-candidate voter model has no counterpart in the traditional BARW models. Notice that, in the suggested BARW models, the evolution into an absorbing state is prevented by the divergency of the branching rate in the limit $c \to 0$ for sufficiently large system size. Further systematic research is required to clarify the effect of these phenomena.

To summarize, in the present paper we have improved the accuracy of the numerical analysis of the critical transition appearing in the vortex density for the three-candidate voter model when varying the magnitude of cyclic dominance. Recognizing that the dynamics of the vortices and antivortices is similar to a BARW model with a density-dependent particle-antiparticle pair creation (branching) rate, we have contrasted these two systems. According to our comparison, we can state that the power law behavior of the vortex density is reproducible with a suitable choice of the pair creation mechanism. The same is not true for the behavior of fluctuations, which seems to be quite different in the two models. This discrepancy is a motivation to seek further extensions of BARW models, since this approach seems to be very useful in the investigation of the self-organizing, three-color domain structures.

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[10] For a recent textbook, see P. Morris, Introduction to Game Theory (Springer-Verlag, New York, 1994).


